Piero Pagliani

Approximation and Pretopologies
Approximation Spaces

- given a set $U$
- given a relation $R$ which structures $U$

$\langle \varnothing(U), lR, uR \rangle$

- $R$ is an equivalence relation
- $lR$ is an interior operator of a 0-dimensional topological space
- $uR$ is a closure operator of a 0-dimensional topological space
Approximation and Pretopologies

First part:
Observations and Pre-topological Approximation Operators

Second part:
Dynamic Property Systems and Pre-topological Approximation Operators

Third part:
Formal Topological Systems and Pre-topological Approximation Operators
The basic ingredients to represent objects through properties:
- a map from **objects** to properties
- a map from **properties** to objects

A set of properties

\[ G \xrightarrow{M} G \]

\[ f \quad g \]

result of the factorisation through \( f \) and \( g \)

A set of objects

Same set of objects

**Gegenstand** \(\rightarrow\) **Objekt**
An “ideal” situation: $f$ is an injective function.
The best model of a horse is a horse?

Modelling means abstraction
Abstraction means loosing something

Horse-ness property

(it must be injective)
An almost ideal situation: \( f \) is a surjective function

\[ M \rightarrow M \]

\[ 1_M \]

\[ G \rightarrow f \]

\[ \text{SECTION} \]

Objects

Observable properties

Sorts (fibres, stalks)
The basic picture

Objects and properties are linked by means of an arbitrary fulfillment relation

\[ \models \subseteq G \times M \]

We call \( \langle G, M, \models \rangle \) a Property System or a Basic Pair
Observations, Property Systems and Attribute Systems

<table>
<thead>
<tr>
<th>R</th>
<th>$p_1 p_2 p_3 p_4$</th>
<th>R</th>
<th>$a_1 a_2 a_3 a_4 a_5$</th>
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<tbody>
<tr>
<td>a</td>
<td>1 0 1 1</td>
<td>a</td>
<td>3 b 1/3 6.5 0</td>
</tr>
<tr>
<td>b</td>
<td>0 0 1 1</td>
<td>b</td>
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</tr>
<tr>
<td>c</td>
<td>0 0 1 1</td>
<td>c</td>
<td>1 a 2 7 1</td>
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<tr>
<td>d</td>
<td>1 0 0 1</td>
<td>d</td>
<td>1 c 1/2 3.3 1</td>
</tr>
<tr>
<td>e</td>
<td>0 1 1 0</td>
<td>e</td>
<td>3 b 1 6 0</td>
</tr>
</tbody>
</table>

*a Property system*  
*an Attribute system*

**Observation systems**
Observation and substantiation

$m \in \text{obs}(g)$ if and only if $g \models m$ if and only if $g \in \text{sub}(m)$
A subset of objects $O$ fulfils a subset of properties $P$

A subset of properties $P'$ describes a subset of objects $O'$

Usually, $\text{sub}(\text{obs}(O)) \supseteq O$

However, is $\text{sub}(\text{obs}(O)) \supseteq O$ an approximation of $O$ by means of the set of properties $\text{obs}(O)$?

No, because $\text{obs}(\text{sub}(\text{obs}(O))) \supseteq \text{obs}(O)$. That is, $O'$ is defined by means of properties which do not have any “essential” relationship with the set $\text{obs}(O)$.
Approximations and best approximations. 2

How to define two operators \( \varphi \) and \( \psi \) by means of \( obs \) and \( sub \) (i.e. \( \models \)) such that the following holds:

\[
\varphi(G) \subseteq (\supseteq) la (ua) \Rightarrow \varphi(M) \Rightarrow \varphi(G)
\]
Towards a solution. 1
Operators must fulfil maximality (minimality), i. e. limit, properties
Towards a solution. 2

Maximality (minimality), implies isotonicity

If the approximations are not defined by means of isotonic operators, then the results could be incomparable: we could hardly speak of a “best” approximation.
Continuity and approximations: 1

Let \( \psi \) be isotonic and \( Y \subseteq G \).

If \( \bigcap \psi^{-1}(\uparrow Y) \) belongs to \( \psi^{-1}(\uparrow Y) \), then

it is the least \( X \) s. t. \( \psi(X) \supseteq Y \)

i. e. the best approximation from above of \( Y \) via \( \psi \).

But if this is the case, then

\( \psi^{-1}(\uparrow Y) = \uparrow X \), for \( X = \bigcap \psi^{-1}(\uparrow Y) \)

that is, the pre-image of a principal filter is a principal filter

Otherwise stated: \( \psi \) is lower-residuated

Dually, if \( \psi \) is upper-residuated, then

the pre-image of a principal ideal is a principal ideal

and \( \bigcup \psi^{-1}(\downarrow Y) \) is the largest \( X \) such that

\( \psi(X) \subseteq Y \)

i. e. the best approximation from below of \( Y \) via \( \psi \).
If $\psi$ is isotonic and lower-residuated, then set

$$\psi^- = \min(\psi^{-1}(\uparrow Y))$$

$\psi^-$ is called the *lower residual* of $\psi$.

If $\psi$ is isotonic and upper-residuated, then set

$$\psi^+ = \max(\psi^{-1}(\downarrow Y))$$

$\psi^+$ is called the *upper residual* of $\psi$. 
Continuity and approximations: 3

\[ \varphi \leq (\sup) \leq \psi \]

\[ \varphi (G) \subseteq (\sup) \subseteq \psi (G) \]

If \( \psi \) is isotonic and lower residuated, set \( \varphi = \psi^- \).

Then \( \psi\varphi(Y) \supseteq Y \)

and for any other \( X \) such that \( \psi(X) \supseteq Y \),

\( \varphi(Y) \subseteq X \)

If \( \psi \) is isotonic and upper-residuated, set \( \varphi = \psi^+ \).

Then \( \psi\varphi(Y) \subseteq Y \)

and for any other \( X \) such that \( \psi(X) \subseteq Y \),

\( \varphi(Y) \supseteq X \)
Let $O$ and $O'$ be two ordered sets.

Let $\sigma: O \to O'$ and $\iota: O' \to O$, such that

$$\forall p \in O, \forall p' \in O', \iota(p') \leq p \text{ if and only if } p' \leq \sigma(p)$$

Then $\iota$ is called the **lower adjoint** of $\sigma$
and $\sigma$ is called the **upper adjoint** of $\iota$

$$O' \vdash \sigma | O$$

The pair $\langle \sigma, \iota \rangle$ is called a **Galois adjunction** or an **axiality** and the following holds:

$\sigma \iota$ is a **closure** operator
$
\iota \sigma$ is an **interior** operator

If $O' \vdash \sigma | O^{op}$,
then $\langle \sigma, \iota \rangle$ is called a **Galois connection** or a **polarity**.

In this case, both $\sigma \iota$ and $\iota \sigma$ are closure operators.
Approximations and Galois Adjunctions. 2

If $\phi = \psi^-$, then $\psi = \phi^+$ and we have

$\phi(G) \triangleright \psi \triangleright M$

Problem:
How one can define a Galois adjunction on an Observation System, in a natural way?
Basic perception constructors. 1

Let us extend the constructors \( \textit{obs} \) and \( \textit{sub} \) from tokens to types

\[
\langle e \rangle : \wp(M) \rightarrow \wp(G) ; \langle e \rangle(Y) = \{g \in G : \exists m(m \in Y \land g \in \text{sub}(m))\}
\]
existent extension of \( \textit{sub} \)

\[
[[e]] : \wp(M) \rightarrow \wp(G) ; [[e]](Y) = \{g \in G : \forall m(m \in Y \Rightarrow g \in \text{sub}(m))\}
\]
universal extension of \( \textit{sub} \)

\[
[e] : \wp(M) \rightarrow \wp(G) ; [e](Y) = \{g \in G : \forall m(g \in \text{sub}(m) \Rightarrow m \in Y)\}
\]
co-universal extension of \( \textit{sub} \)

\[
\langle i \rangle : \wp(G) \rightarrow \wp(M) ; \langle i \rangle(X) = \{m \in M : \exists g(g \in X \land m \in \text{obs}(g))\}
\]
existent extension of \( \textit{obs} \)

\[
[[i]] : \wp(G) \rightarrow \wp(M) ; [[i]](X) = \{m \in M : \forall g(g \in X \Rightarrow m \in \text{obs}(g))\}
\]
universal extension of \( \textit{obs} \)

\[
[i] : \wp(G) \rightarrow \wp(M) ; [i](X) = \{m \in M : \forall g(m \in \text{obs}(g) \Rightarrow g \in X)\}
\]
co-universal extension of \( \textit{obs} \)
Basic perception constructors. 2

\[ \langle i \rangle: \varphi(G) \rightarrow \varphi(M) ; \langle i \rangle(X)=\{m \in M : \exists g (g \in X \land m \in \text{obs}(g))\} \]
existential extension of \text{obs}

\[ [[i]]: \varphi(G) \rightarrow \varphi(M) ; [[i]](X)=\{m \in M : \forall g (g \in X \Rightarrow m \in \text{obs}(g))\} \]
universal extension of \text{obs}

\[ [i]: \varphi(G) \rightarrow \varphi(M) ; [i](X)=\{m \in M : \forall g (m \in \text{obs}(g) \Rightarrow g \in X)\} \]
co-universal extension of \text{obs}
Intuitive and modal reading of the basic perception constructors

- **Possible** for elements of $X$ to fulfil $b$
- **Necessary** to be in $X$
- **Sufficient** to be in $X$

- **At least one**
  - there is an example of elements of $X$ which fulfils $b$

- **At most all**
  - there are not examples of elements outside $X$ which fulfill $b$

- **At least all**
  - there are not examples of elements in $X$ which do not fulfill $b$

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Università di Milano Bicocca maggio 2009
Adjunction properties of the basic perception constructors

\[ \langle i \rangle (X) \subseteq Y \text{ if and only if } X \subseteq [e](Y) \]

\[ \langle e \rangle (Y) \subseteq X \text{ if and only if } Y \subseteq [i](X) \]

\[ [[e]](Y) \subseteq X \text{ if and only if } [[i]](X) \subseteq Y \]

for all \( X \subseteq G \), for all \( Y \subseteq M \)

\[ \langle i \rangle \dashv \ [e] \]

\[ \langle e \rangle \dashv \ [i] \]

\[ [\bigwedge] \text{ is an } \textbf{interior} \text{ operator} \]

\[ [\bigvee] \langle i \rangle \text{ is an } \textbf{closure} \text{ operator} \]

\[ \text{anti-multiplicative} \]

\[ \text{anti-isotonic} \]

\[ [[e]] \dashv \text{op} \ [i] \]

\[ \text{anti-multiplicative} \]

\[ \text{anti-isotonic} \]
Combining adjoint constructors: perception operators

<table>
<thead>
<tr>
<th>modal-style operators</th>
<th>intent-extent operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Qualitative Data Analysis</td>
<td>Formal Concept Analysis</td>
</tr>
<tr>
<td>( \text{int} : \wp(G) \to \wp(G) ); ( \text{int}(X) = \langle e \rangle<a href="X">i</a> )</td>
<td>( \text{est} : \wp(G) \to \wp(G) ); ( \text{est}(X) = [[e]]<a href="X">[i]</a> )</td>
</tr>
<tr>
<td>( \text{cl} : \wp(G) \to \wp(G) ); ( \text{cl}(X) = [e] \langle i \rangle(X) )</td>
<td>( \text{ITS} : \wp(M) \to \wp(M) ); ( \text{ITS}(Y) = [[i]]<a href="Y">[e]</a> )</td>
</tr>
<tr>
<td>( \text{C} : \wp(M) \to \wp(M) ); ( \text{C}(Y) = \langle i \rangle<a href="Y">e</a> )</td>
<td>( \text{A} : \wp(M) \to \wp(M) ); ( \text{A}(Y) = [i] \langle e \rangle(Y) )</td>
</tr>
</tbody>
</table>

Diagram:

\[ G \]

\[ M \]

- \( \text{int}\{a, a'\} \)
- \( \text{cl}\{a, a'\} \)
- \( \text{est}\{a, a'\} \)
- \( \text{A}\{b', b''\} \)
- \( \text{ITS}\{b', b''\} \)
- \( \text{C}\{b'', b'''\} \)
But what do \textit{int}, \textit{cl}, \textit{C} and \textit{A} represent, actually?

Preliminary question: what does \textit{M} represent?

Each element \textit{m} of \textit{M} represents the set \langle e \rangle (m) of elements of \textit{G} which fulfil property \textit{m}. Hence \textit{m} collects elements of \textit{G} by means of the \textit{nearness relation} “to fulfil \textit{m}”.

\textit{m} is a \textit{formal neighbourhood} a proxy for a subset of \textit{G}

\begin{itemize}
  \item \textit{[i]}(X) is the set of formal neighbourhoods \textit{m} “included” in \textit{X}, i. e. such that \langle e \rangle (m) \subseteq X
  \item \langle e \rangle [i](X) is the extension of such neighbourhoods.
\end{itemize}

Therefore: \langle e \rangle [i](X) denotes in a \textit{formal} way the \textit{interior} of \textit{X}.

\begin{itemize}
  \item \langle i \rangle (X) is the set of formal neighbourhoods \textit{m} with non void “intersection” with \textit{X}. i. e. such that \textit{X} \cap \langle e \rangle (m) \neq \emptyset
  \item [e] \langle i \rangle (X) is the extension of all and only such neighbourhoods.
\end{itemize}

Therefore: [e] \langle i \rangle (X) denotes in a \textit{formal} way the \textit{closure} of \textit{X}.

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(Pre) topological properties of the perception operators

We obtain G. Sambin’s results on Formal Topology (point-free topology)

Straightforwardly from the adjunction properties between the basic constructors

FORMAL

closed

closed

open

open

\forall \exists : [e]<i> \equiv \textit{cl}

\forall \exists : [i]<e> \equiv A : \forall \exists

CONCRETE

FORMAL

\exists : <e>

\exists : <i>

\forall \exists : [e][i] \equiv \text{int}

\forall \exists : [i][e] \equiv C : \exists \forall

\forall \exists : [e] \equiv \text{symmetric}

\forall \exists : [i] \equiv \text{symmetric}

\forall : [e]

\forall : [i]
Algebraic properties of the perception operators

Given the following set of fixpoints of the operators

1. $\Omega_{\text{int}}(P) = \{X \subseteq G : \text{int}(X) = X\}; \quad \Gamma_{\text{cl}}(P) = \{X \subseteq G : \text{cl}(X) = X\}$;
2. $\Gamma_{\text{est}}(P) = \{X \subseteq G : \text{est}(X) = X\}; \quad \Omega_{\mathcal{A}}(P) = \{Y \subseteq M : \mathcal{A}(Y) = Y\}$;
3. $\Gamma_{\mathcal{C}}(P) = \{Y \subseteq M : \mathcal{C}(Y) = Y\}; \quad \Gamma_{\mathcal{IT}S}(P) = \{Y \subseteq M : \mathcal{IT}S(Y) = Y\}$.

one obtains the following lattices

1. $\text{Sat}_{\text{int}}(P) = \langle \Omega_{\text{int}}(P), \cup, \wedge, \emptyset, G \rangle$, where $\bigwedge_{i \in I} X_i = \text{int} (\bigcap_{i \in I} X_i)$;
2. $\text{Sat}_{\mathcal{A}}(P) = \langle \Omega_{\mathcal{A}}(P), \vee, \cap, \emptyset, M \rangle$, where $\bigvee_{i \in I} Y_i = \mathcal{A} (\bigcup_{i \in I} Y_i)$;
3. $\text{Sat}_{\text{cl}}(P) = \langle \Gamma_{\text{cl}}(P), \vee, \cap, \emptyset, G \rangle$, where $\bigvee_{i \in I} X_i = \text{cl} (\bigcup_{i \in I} X_i)$;
4. $\text{Sat}_{\mathcal{C}}(P) = \langle \Gamma_{\mathcal{C}}(P), \cup, \wedge, \emptyset, M \rangle$, where $\bigwedge_{i \in I} Y_i = \mathcal{C} (\bigcap_{i \in I} Y_i)$;
5. $\text{Sat}_{\text{est}}(P) = \langle \Gamma_{\text{est}}(P), \cap, \vee, \text{est} (\emptyset), G \rangle$, where $\bigvee_{i \in I} X_i = \text{est} (\bigcup_{i \in I} X_i)$;
6. $\text{Sat}_{\mathcal{IT}S}(P) = \langle \Gamma_{\mathcal{IT}S}(P), \cap, \vee, \mathcal{IT}S (\emptyset), M \rangle$, where $\bigvee_{i \in I} Y_i = \mathcal{IT}S (\bigcup_{i \in I} Y_i)$. 
**Example**

<table>
<thead>
<tr>
<th>P</th>
<th>b</th>
<th>b₁</th>
<th>b₂</th>
<th>b₃</th>
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</thead>
<tbody>
<tr>
<td>a</td>
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<td>0</td>
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</table>

\[ \text{Sat}_{int}(P) \]

\[ \text{Sat}_{el}(P) \]
Approximation properties of perception operators

\[ \langle e \rangle[i](X) \subseteq X \subseteq [e]\langle i \rangle(X) \]

i. e.

\[ \text{int} (X) \subseteq X \subseteq \text{cl} (X) \]

with maximality, respectively, minimality property with respect to inclusion

for all \( X \subseteq G \)

*(immediate from the adjunction properties)*

\[ \text{int} \text{ is a pre-topological lower approximation operator} \]

\[ \text{cl} \text{ is a pre-topological upper approximation operator} \]
Choosing the initial perception act.

Information quanta

J. L. Bell’s approach to Quantum Logic: any object $g$ is perceived together with all the elements which are indiscernible from $g$ in the “field of perception”. Hence any object is a location and it is perceived embedded in a quantum at that location: minimum perceptibilium at that location.

“Orthologic, forcing and the manifestation of attributes”.

We take this phenomenological approach and claim that any object $g$ is perceived together with all the elements which manifest at least all the same properties as $g$: information quantum at $g$.

$g' \in Q_g$ if and only if $g'$ fulfils at least all the properties fulfilled by $g$:

$\langle i \rangle (\{g\}) \subseteq \langle i \rangle (\{g'\})$
Information quantum relation and Information quantum systems

Given a Property System $S$

\[ \langle g, g' \rangle \in R_S \text{ if and only if } g' \in Q_g \]

\( Q(S) = \langle G, G, R_S \rangle \)

Information quantum relation
- reflexive
- transitive

Information quantum relation system
- it is a property system!!

If $\langle g', g \rangle \in R_S$ then $g' \in Q_g$, thus $\langle i \rangle(g) \subseteq \langle i \rangle(g')$.

Hence $g$ is less defined than $g'$ in $S$.

Indeed, $R_S$ is a (anti)specialisation preorder in the usual topological sense.
Adjunction and algebraic properties of $i$-quantum operators

In $Q(S)$ the sets of fixpoints of adjoint $i$-quantum operators coincide

Let $cl$, $int$, $C$ and $A$ be induced by $Q(S)$. Then

\[ cl = \langle i \rangle \quad int = [i] \]
\[ C = [e] \quad A = \langle e \rangle \]

Since $cl$ is lower adjoint, $cl$ is additive, hence topological

Since $int$ is upper adjoint, $int$ is multiplicative, hence topological

Similarly for $A$ and $C
### Example

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</table>

\[
\text{Sat}_{cl}(Q(P))
\]

![Diagram](image)
Topological approximation operators

Let $cl$, $int$, $C$ and $\mathcal{A}$ in $Q(S)$. Then

$\text{cl} = \langle i \rangle$: inverse upper approximation
$\text{cl}(X) = \text{set of elements approximated by some member of } X$

$\text{int} = [i]$: inverse lower approximation
$\text{int}(X) = \text{set of elements approximated just by members of } X$

$C = [e]$: direct lower approximation
$C(X) = \text{set of elements specialised just by members of } X$

$\mathcal{A} = \langle e \rangle$: direct upper approximation
$\mathcal{A}(X) = \text{set of elements specialised by some member of } X$
A particular case

If $S$ is an Attribute System, its nominalisation is a Perception System $\mathcal{M}(S)$ such that:

$$R_{\mathcal{M}(S)}$$ is an equivalence relation

and in $Q(\mathcal{M}(S))$

$$cl(X) = \text{upper approximation of } X$$

$$int(X) = \text{lower approximation of } X$$

in the sense of Pawlak’s classical Rough Set Theory

Indeed, in an Attribute System “to fulfill at least the same properties” is tantamount to saying “to have the same attribute value”, which induces an equivalence relation

The same is true of dichotomic Property Systems
Second part:

*Dynamic Property Systems* 
and 

*Pre-topological Approximation Operators*
• Z. Pawlak

*Approximation Spaces*

\(<U, E>\)

for \(E\) an equivalence relation

• T. Y. Lin & Y. Y. Yao

*Neighbourhood Spaces*

\(<U, R>\)

for \(R\) any binary relation

• A. Skowron & J. Stepaniuk

• P. Pagliani

*Approximation of relations*

\(<U, \{E_i\}_{i \in I}>\)

for \(\{E_i\}_{i \in I}\) a family of equivalence relations

• P. Pagliani

*Dynamic Approximation Spaces*

\(<U, \{R_i\}_{i \in I}>\)

for \(\{R_i\}_{i \in I}\) a family of binary relations
Sources of dynamics

1) Evolution of the observation process over time
2) Change of context
3) 1+2
A layered observation system
Why more than one neighbourhood?

More than 1 context $\rightarrow$ More than 1 neighbourhood

- Context $\alpha$
  - $U'$: applicable criteria (metrics) of type $T$
- Context $\beta$

$U$: object level

$N_x$

Neighbourhood of $x$

More than 1 context

Why more than one neighbourhood?
What if more than one neighbourhood?

More than 1 neighbourhood $\rightarrow$ Neighbourhood families

More neighbourhood families $\rightarrow$ Neighbourhood systems

---

Let $U, U'$ be sets, $X \subseteq U'$, $u \in U$ and $f : U \leftrightarrow U'$ a total function. Then,

1. a neighbourhood map is a total function $n : U \leftrightarrow \varphi(\varphi(U'))$;

2. (i) $n(u)$ is called a neighbourhood family of $u$; (ii) if $N \in n(u)$, then $N$ is called a neighbourhood of $u$; (iii) if $u' \in N \in n(u)$, then $u'$ is called a neighbour of $u$; (iv) the family $\mathcal{N}(U) = \{n(x) : x \in U\}$ is called a (generalised) neighbourhood system; (v) the pair $\langle U, \mathcal{N}(U) \rangle$ is called a (concrete) neighbourhood space.
Core maps and vicinity maps

3. if \( G(X) = \{x : X \in n(x)\} \), then \( G \) is called the core map induced by \( \mathcal{N}(U) \);

4. if \( F(X) = -G(-X) = \{x : -X \notin n(x)\} \), then \( F \) is called the vicinity map induced by \( \mathcal{N}(U) \);

5. the set \( F(X) \cap -G(X) = \{x : \forall N \in n(x)(N \cap X \neq \emptyset \neq N \cap -X)\} \) is called the boundary of \( X \) and denoted by \( \partial(X) \);

Let us denote \( n(x) \) with \( \mathcal{N}_x \)
Sectioning topological systems

For the sake of simplicity, let $U' = U$ and let $f$ be the identity function. One can distinguish the following properties, which define a topological space, for any $x \in U$, $A \subseteq U$:

1. $U \in \mathcal{N}_x$
2. $\emptyset \notin \mathcal{N}_x$
3. If $x \in G(A)$ then $G(A) \in \mathcal{N}_x$
4. $x \in N$, for all $N \in \mathcal{N}_x$
5. If $N \in \mathcal{N}_x$ and $N \subseteq N'$ then $N' \in \mathcal{N}_x$
6. If $N$, $N' \in \mathcal{N}_x$ then $N \cap N' \in \mathcal{N}_x$
7. There is an $N \neq \emptyset$ such that $\mathcal{N}_x = \uparrow N$
Conditions and properties of neighbourhood systems

Conditions 0, 1, Id, N1, N2 and N3 characterise **topological spaces**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equivalent properties of $G$</th>
<th>Equivalent properties of $F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G(U) = U$ (normality)</td>
<td>$F(\emptyset) = \emptyset$ (normality)</td>
</tr>
<tr>
<td>0</td>
<td>$G(\emptyset) = \emptyset$ (co-normality)</td>
<td>$F(U) = U$ (co-normality)</td>
</tr>
<tr>
<td>Id</td>
<td>$G(X) \subseteq G(G(X))$</td>
<td>$F(F(X)) \subseteq F(X)$</td>
</tr>
<tr>
<td>N1</td>
<td>$G(X) \subseteq X$ (deflation)</td>
<td>$X \subseteq F(X)$ (inflation)</td>
</tr>
<tr>
<td>N2</td>
<td>$X \subseteq Y \Rightarrow G(X) \subseteq G(Y)$ (isotonicity)</td>
<td>$X \subseteq Y \Rightarrow F(X) \subseteq F(Y)$</td>
</tr>
<tr>
<td></td>
<td>$G(X \cap Y) \subseteq G(X) \cap G(Y)$</td>
<td>$F(X \cup Y) \supseteq F(X) \cup F(Y)$</td>
</tr>
<tr>
<td>N3</td>
<td>$G(X \cap Y) \supseteq G(X) \cap G(Y)$</td>
<td>$F(X \cup Y) \subseteq F(X) \cup F(Y)$</td>
</tr>
</tbody>
</table>
Neighbourhood systems and systems of relations.

<table>
<thead>
<tr>
<th>The elements of $\mathcal{N}(U)$ satisfy:</th>
<th>$\mathcal{N}(U)$ is of type:</th>
<th>The elements of $\mathcal{N}(U)$ are:</th>
<th>$\mathcal{N}(U)$ is induced by:</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 1, N1</td>
<td>$\mathcal{N}_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0, 1, N1, Id</td>
<td>$\mathcal{N}_{1Id}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0, 1, N1, N2</td>
<td>$\mathcal{N}_2$</td>
<td>proper $\subseteq$-filters</td>
<td>a system of reflexive relations</td>
</tr>
<tr>
<td>0, 1, N1, N2, $\tau$</td>
<td>$\mathcal{N}_{2Id}$</td>
<td>”</td>
<td>a system of preorders</td>
</tr>
<tr>
<td>0, 1, N1, N2, N3</td>
<td>$\mathcal{N}_3$</td>
<td>”</td>
<td></td>
</tr>
<tr>
<td>0, 1, N1, N2, N3, $\tau$</td>
<td>$\mathcal{N}_{3Id}$</td>
<td>”</td>
<td></td>
</tr>
<tr>
<td>0, 1, N1, N2, N3, N4</td>
<td>$\mathcal{N}_4$</td>
<td>principal $\subseteq$-filters</td>
<td>a single reflexive relation</td>
</tr>
<tr>
<td>0, 1, N1, N2, N3, N4, $\tau$</td>
<td>$\mathcal{N}_{4Id}$</td>
<td>”</td>
<td>a single preorder</td>
</tr>
</tbody>
</table>
Expansion and contraction processes

An expansion is a process that applied to a set $X$ collects all the elements of $X$ plus those elements that, under some point of view, are connected with them (if any). Therefore such a process is an increasing map $f$ between subsets of $U$:

Definition 2.3. Let $U$ be a set. An expansion process is any map $f : \varphi(U) \longmapsto \varphi(U)$ such that for any $X \subseteq U$, $X \subseteq f(X)$.

Dually, we can think of a process of erosion that cuts down some connections between elements of $U$, just leaving the elements from a subset $X$ that are strictly connected. We call such a process a ”contraction”.

Definition 2.4. Let $U$ be a set. A contraction process is any map $g : \varphi(U) \longmapsto \varphi(U)$ such that for any $X \subseteq U$, $g(X) \subseteq X$. 
Expansion and contraction are not necessarily isotonic
**Pre-topological spaces and neighbourhood systems**

- A pre-topological space is a triple $\langle U, \varepsilon, \kappa \rangle$ such that:
  (i) $U$ is a set, (ii) $\varepsilon: \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ is an expansion map such that $\varepsilon(\emptyset) = \emptyset$, (iii) $\kappa: \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ is a contraction map dual of $\varepsilon$.

<table>
<thead>
<tr>
<th>$\langle U, \varepsilon, \kappa \rangle$ is said of type:</th>
<th>if and only if it satisfies:</th>
<th>if $N^\kappa(U)$ is of type:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{V}_{Id}$</td>
<td>$\kappa(\kappa(X)) = \kappa(X)$</td>
<td>$N_{1Id}$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon(\varepsilon(X)) = \varepsilon(X)$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{V}_I$</td>
<td>$X \subseteq Y \Rightarrow \kappa(X) \subseteq \kappa(Y)$ [$\varepsilon(X) \subseteq \varepsilon(Y)$]</td>
<td>$N_2$</td>
</tr>
<tr>
<td>$\mathcal{V}_D$</td>
<td>$\varepsilon(X \cup Y) = \varepsilon(X) \cup \varepsilon(Y)$ [$\kappa(X \cap Y) = \kappa(X) \cap \kappa(Y)$]</td>
<td>$N_3$</td>
</tr>
<tr>
<td>$\mathcal{V}_{Cl}$</td>
<td>$\varepsilon [\kappa]$ is a closure [interior] operator</td>
<td>$N_{2Id}$</td>
</tr>
<tr>
<td>$\mathcal{V}_S$</td>
<td>$\varepsilon(X) = \bigcup_{x \in X} \varepsilon({x})$</td>
<td>$N_4$</td>
</tr>
</tbody>
</table>

- The core map (vicinity map) induced by a neighbourhood system of type (at least) $N_1$ is a contraction map (an expansion map)
Dynamic Approximation Spaces

\[ \langle U, \{R_i\}_{1 \leq i \leq n}, \kappa^m, \varepsilon^m \rangle \]
for \( \{R_i\}_{1 \leq i \leq n} \) a family of binary relations, \( 1 \leq m \leq n \)

\( n \) use case involving the expansion process,
and \( n \) use cases involving the contraction process

(Contraction): We say that \( x \in \kappa^m(A) \), for \( 1 \leq m \leq n \), if every \( y \) such that \( < x, y > \in R_i \) belongs to \( A \), at least in \( m \) cases. Otherwise stated: \( x \in \kappa^m(A) \) if \( R_{1\leq i \leq n}(x) \subseteq A \) for at least \( m \) indices. So, for instance, assume \( n = 3 \), then \( x \in \kappa^2(A) \) if \( R_1(x) \subseteq A \) and \( R_2(x) \subseteq A \), or if \( R_1(x) \subseteq A \) and \( R_3(x) \subseteq A \), or if \( R_2(x) \subseteq A \) and \( R_3(x) \subseteq A \) (i.e. if \( R_1(x) \cup R_2(x) \subseteq A \), or \( R_1(x) \cup R_3(x) \subseteq A \), or \( R_2(x) \cup R_3(x) \subseteq A \)).

(Expansion): We say that \( x \in \varepsilon^m(A) \), for \( 1 \leq m \leq n \), if \( A \) contains at least a \( y \) such that \( < x, y > \in R_i \) in at least \( n + 1 - m \) cases. Otherwise stated: \( x \in \varepsilon^m(A) \) if \( R_{1\leq i \leq n}(x) \cap A \neq \emptyset \) for at least \( n + 1 - m \) indices. So, for instance, assume \( n = 3 \), then \( x \in \varepsilon^3(A) \) if \( R_1(x) \cap A \neq \emptyset \), or \( R_2(x) \cap A \neq \emptyset \), or \( R_3(x) \cap A \neq \emptyset \) (i.e. if \( (R_1(x) \cup R_2(x) \cup R_3(x)) \cap A \neq \emptyset \)).
Let $U$ be a set and $\mathcal{R} = \{R_i\}_{1 \leq i \leq n}$ a system of $n$ binary relations on $U$. For $1 \leq m \leq n$, let $\Gamma$ be the family of combinations of $m$ elements out of a set of $n$ elements, $\gamma$ a combination from $\Gamma$. Then let us set:

(1) $\varepsilon^m : \varnothing(U) \longrightarrow \varnothing(U); \varepsilon^m(A) = \{x \in U : \forall F(F \in F_x^m \Rightarrow F \cap A \neq \emptyset)\}$;

(2) $\kappa^m : \varnothing(U) \longrightarrow \varnothing(U); \kappa^m(A) = \{x \in U : \exists F(F \in F_x^m \land F \subseteq A)\}$,

where: $F_x^m$ is the (order) filter induced by the basis $B_x^m$, (b) $B_x^m = \{X_\gamma : X_\gamma = \bigcup_{l \in \gamma} R_l(x)\}$. 

Pre-topological spaces and Dynamic Spaces
<table>
<thead>
<tr>
<th>$m$</th>
<th>$\Gamma$</th>
<th>$B_x^m$</th>
<th>$B_a^m$</th>
<th>$B_b^m$</th>
<th>$B_c^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${{1}, {2}, {3}}$</td>
<td>${R_1(x), R_2(x), R_3(x)}$</td>
<td>${\emptyset}$</td>
<td>${\emptyset}$</td>
<td>${\emptyset}$</td>
</tr>
<tr>
<td>2</td>
<td>${{1} \cup {2}}, {{1} \cup {3}}, {{2} \cup {3}}$</td>
<td>${R_1(x) \cup R_2(x), R_1(x) \cup R_3(x), R_2(x) \cup R_3(x)}$</td>
<td>${\emptyset}$</td>
<td>${\emptyset}$</td>
<td>${\emptyset}$</td>
</tr>
<tr>
<td>3</td>
<td>${{1} \cup {2} \cup {3}}$</td>
<td>${R_1(x) \cup R_2(x) \cup R_3(x)}$</td>
<td>${\emptyset}$</td>
<td>${\emptyset}$</td>
<td>${\emptyset}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mathcal{F}_x^m$</th>
<th>$\mathcal{F}_a^m$</th>
<th>$\mathcal{F}_b^m$</th>
<th>$\mathcal{F}_c^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{F}_x^1$</td>
<td>${{a, b}, {a, c}, {a, b, c}}$</td>
<td>${{b}, {a, b}, {b, c}, {a, b, c}}$</td>
<td>${{c}, {a, c}, {b, c}, {a, b, c}}$</td>
</tr>
<tr>
<td>$\mathcal{F}_x^2$</td>
<td>${{a, c}, {a, b, c}}$</td>
<td>${{b}, {a, b}, {b, c}, {a, b, c}}$</td>
<td>${{c}, {a, c}, {b, c}, {a, b, c}}$</td>
</tr>
<tr>
<td>$\mathcal{F}_x^3$</td>
<td>${{a, b, c}}$</td>
<td>${{a, b}, {a, b, c}}$</td>
<td>${{c}, {a, c}, {b, c}, {a, b, c}}$</td>
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</table>

<table>
<thead>
<tr>
<th>$\varepsilon^m$</th>
<th>$\emptyset$</th>
<th>${a}$</th>
<th>${b}$</th>
<th>${c}$</th>
<th>${a, b}$</th>
<th>${a, c}$</th>
<th>${b, c}$</th>
<th>${a, b, c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon^1$</td>
<td>$\emptyset$</td>
<td>${a}$</td>
<td>${b}$</td>
<td>${c}$</td>
<td>${a, b}$</td>
<td>${a, c}$</td>
<td>${b, c}$</td>
<td>${a, b, c}$</td>
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<td>${a, c}$</td>
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<td>${a, c}$</td>
<td>${b, c}$</td>
<td>${a, b, c}$</td>
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<td>$\varepsilon^3$</td>
<td>$\emptyset$</td>
<td>${a, b}$</td>
<td>${a, b}$</td>
<td>${a, c}$</td>
<td>${a, b}$</td>
<td>${a, c}$</td>
<td>${b, c}$</td>
<td>${a, b, c}$</td>
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</table>

<table>
<thead>
<tr>
<th>$\kappa^m$</th>
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<th>${b}$</th>
<th>${c}$</th>
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<th>${a, c}$</th>
<th>${b, c}$</th>
<th>${a, b, c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa^1$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${b}$</td>
<td>${c}$</td>
<td>${a, b}$</td>
<td>${a, c}$</td>
<td>${b, c}$</td>
<td>${a, b, c}$</td>
</tr>
<tr>
<td>$\kappa^2$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${b}$</td>
<td>${c}$</td>
<td>${b}$</td>
<td>${a, c}$</td>
<td>${b, c}$</td>
<td>${a, b, c}$</td>
</tr>
<tr>
<td>$\kappa^3$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>${c}$</td>
<td>${b}$</td>
<td>${c}$</td>
<td>${c}$</td>
<td>${a, b, c}$</td>
</tr>
</tbody>
</table>
Associating systems of relations with pre-topological spaces

Let $U$ be a set and let $\mathcal{R} = \{R_i\}_{i \in I}$ be a system of reflexive relations on $U$. Then the pre-topological space induced by the basis $\mathcal{B}^m(U) = \{\mathcal{B}_x^m\}_{x \in U}$ is said to be $m$–associated with the system $\mathcal{R}$ and denoted by $\mathbf{P}^m(\mathcal{R}) = \langle U, \varepsilon^m, \kappa^m \rangle$. In particular if $\mathcal{R} = \{R\}$, then the pre-topological space induced by the basis $\{R(x)\}_{x \in U}$ is said to be associated with the relation $R$ and denoted by $\mathbf{P}(R) = \langle U, \varepsilon^R, \kappa^R \rangle$.

Theorem 3.1. Let $\langle U, \{R_i\}_{i \in I} \rangle$ a Dynamic Space such that $\mathcal{R} = \{R_i\}_{i \in I}$ is a family of preorder relations on $U$. Then: (1) in the pre-topological space $\mathbf{P}^1(\mathcal{R}) = \langle U, \varepsilon^1, \kappa^1 \rangle$ the operators $\varepsilon^1$ and $\kappa^1$ are isotonic and idempotent; (2) the family $\{\uparrow \mathcal{B}_x^1\}_{x \in U}$ is a neighbourhood system of type $\mathcal{N}_{2Id}$.

Theorem 3.2. Let $\mathbf{P} = \langle U, \varepsilon^R, \kappa^R \rangle$ be a pre-topological space associated with a reflexive binary relation $R \subseteq U \times U$. Then $\mathbf{P}$ is a topological space if and only if $R$ is transitive.
Piero Pagliani

Approximation and pre-topologies

Third part

Formal Topological Systems
and
Pre-topological Approximation Operators
Concrete and formal pre-topology

Let $P=\langle G, M, R \rangle$, be any property system, for $R \subseteq G \times M$ any binary relation.

We remind that

$$\Omega_{int}(G) = \{ X \subseteq G : \text{int}(X) = X \}$$

is the family of concrete open subsets of $P$

$$\Omega_{\mathcal{A}}(M) = \{ Y \subseteq M : \mathcal{A}(X) = X \}$$

is the family of formal open subsets of $P$

$$\text{Sat}_{int}(G) = \langle \Omega_{int} (G), \cup, \wedge, \emptyset, G \rangle$$

where $\wedge_{i \in I} X_i = \text{int}(\bigcap_{i \in I} X_i)$ is a complete lattice

$$\text{Sat}_{\mathcal{A}}(M) = \langle \Omega_{\mathcal{A}} (M), \vee, \cap, \emptyset, M \rangle$$

where $\vee_{i \in I} X_i = \mathcal{A}(\bigcup_{i \in I} X_i)$ is a complete lattice

Piero Pagliani

Università di Milano Bicocca maggio 2009
A formal semi-covering relation between neighbourhoods

Let \( <G, M, R> \) be a basic pair. Then, for any \( b \in M \) and \( Y, Y' \subseteq M \), a formal semi-cover relation \( \blacktriangleleft \) is defined as follows:

(basis) \( b \blacktriangleleft Y \) iff \( b \in A(Y) \),  
(step) \( Y \blacktriangleleft Y' \) iff \( \forall y \in Y, y \blacktriangleleft Y' \)

From this definition, it follows:

\[
\frac{b \in Y}{b \blacktriangleleft Y}\quad \text{reflexivity}
\]

\[
\frac{b \blacktriangleleft Y \quad Y \blacktriangleleft Y'}{b \blacktriangleleft Y'}\quad \text{transitivity}
\]

\[
Y \blacktriangleleft Y\quad \text{identity}
\]
Combination of formal neighbourhoods and semi-covering relation

Properties may be combined by means of a formal meet operation that we call “fusion” and denote with “•”

The operation “•” is inherited by subsets of properties in the following way:

\[ X \cdot Y = \{ x \cdot y : x \in X \land y \in Y \} \] (point-meet)

Formal opens inherit the operation “•” in the following way:

\[ X \cdot Y = A(X \cdot Y) \]
Properties which can be added to a semi-cover

For all $Y, Y' \in \Omega_{A}(M)$, and $b, b' \in M$

\[
\frac{b \blacktriangleleft Y}{b \cdot b' \blacktriangleleft Y} \quad \text{left}
\]

\[
\frac{b \blacktriangleleft Y \ b \blacktriangleleft Y'}{b \blacktriangleleft Y \cdot Y'} \quad \text{right}
\]

\[
\frac{b \blacktriangleleft Y \ b' \blacktriangleleft Y'}{b \cdot b' \blacktriangleleft Y \cdot Y'} \quad \text{stability}
\]
Property systems and formal systems

- Let \( \langle G, M, R \rangle \) be a basic pair (i.e. property system), let \( \langle M, \cdot, 1 \rangle \) be commutative monoid, set \( \bot = \{ m \in M : R(m) = \emptyset \} \). Then: \( \langle M, \cdot, 1, \triangledown, \bot \rangle \) is called a semi-topological formal system induced by the basic pair \( \langle G, M, R \rangle \)

- A semi-topological formal system in which stability holds, is called a pre-topological formal system, and \( \triangledown \) is called a precover

- A semi-topological formal system in which left and right hold, is called a topological formal system, and \( \triangledown \) is called a cover.

- If \( \langle M, \cdot, 1, \triangledown, \bot \rangle \) is a topological formal system, then \( \langle \Omega \mathcal{A}(M), \cdot, \lor, M, \mathcal{A}(\bot) \rangle \), for \( \bot \) any subset of \( M \), is a complete lattice with complete distributivity and ordering \( \subseteq \).

- In any semi-topological formal system in which \( \cdot \) is idempotent and stability holds, right is derivable.

- Any semi-topological formal system in which \( \cdot \) is idempotent and stability and left hold, is a topological formal system.
A bridge between concrete neighborhood systems and formal systems

- \( \langle A, \varnothing(A), R \rangle \) is called a basic neighborhood pair and, therefore,
  - \( \langle \varnothing(A), \cap, A, \blacklozenge, \bot \rangle \) will be called a formal neighborhood system

Since \( \cap \) is idempotent, any formal neighborhood system in which stability and left hold, is a topological formal system.
Towards the main connection between concrete pre-topological systems and formal systems

Let $\langle \varphi(A), \cap, A, \blacktriangleleft, \perp \rangle$ be induced by $\langle A, \varphi(A), R \rangle$. Then:

a) If $\{R(x) : x \in A\}$ fulfils $\mathbf{N3}$, then right holds. That is:

   for all $X, X' \in \varphi(A), x \in A$ and $Y, Y' \subseteq \varphi(A)$,
   
   $\left( X, X' \in R(x) \Rightarrow X \cap X' \in R(x) \right) \Rightarrow (X \blacktriangleleft Y$ and $X \blacktriangleleft Y' \Rightarrow X \blacktriangleleft Y \cdot Y')$

b) $\{R(x) : x \in A\}$ fulfils $\mathbf{N2}$ if and only if left holds. That is:

   for all $X, X' \in \varphi(A), x \in A$ and $Y \subseteq \varphi(A)$,
   
   $\left( X \in R(x) \& X \subseteq X' \Rightarrow X' \in R(x) \right) \Longleftrightarrow (X \blacktriangleleft Y \Rightarrow X \cdot X' \blacktriangleleft Y)$
Main connection between concrete pre-topological systems and formal systems

Let \( \langle \varnothing(A), \cap, A, \blacktriangle, \perp \rangle \) be induced by \( \langle A, \varnothing(A), R \rangle \). Then, for all \( X \in \varnothing(A) \),
\[
G(X) = \text{int}(X)
\]
if and only if \( \{R(x): x \in A\} \) is a neighborhood system of type \( N_{2Id} \).

A formal neighborhood system \( \langle \varnothing(A), \cap, A, \blacktriangle, \perp \rangle \) such that for all \( X \in \varnothing(A) \),
\[
G(x) = \text{int}(X),
\]
will be called a pseudo-topological formal system.
From concrete pre-topological systems to concrete topological systems, through formal systems

Let \( \langle \emptyset(A), \cap, A, \downarrow, \bot \rangle \) be a pseudo-topological formal system. Then:

\[
\langle \Omega_{\mathcal{A}}(\emptyset(A)), \cap \rangle
\]

is a meet semilattice with ordering \( \subseteq \).

However, in general \( \cap \) does not distribute over \( \lor \) in \( \text{Sat}_{\mathcal{A}}(\emptyset(A)) \).

To obtain distributivity of \( \cap \) over \( \lor \), we need stability:

In any pseudo-topological formal system, if stability holds, then

\[
\langle \Omega_{\mathcal{A}}(\emptyset(A)), \cap, \lor \rangle
\]

is a complete distributive lattice, hence a Heyting algebra.

But this is not a surprise, because:

In any pseudo-topological formal system \( \text{N2} \) holds, by definition. Hence left holds, too. Moreover, the monoidal operation is \( \cap \), which is idempotent. It follows that if we add stability we obtain right.

But we know that a pre-topological formal system in which right and left hold, is a topological formal system.
7.1.1 A sample formal neighborhood system

Consider the neighborhood system induced by the basic pair \( \langle U, \varphi(U), R \rangle \), where \( U = \{a, b, c\} \) and \( R(a) = \{\{a, b\}, \{a, c\}, U\}, R(b) = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}, U\}, R(c) = \{\{c\}, \{a, c\}, U\} \). It induces the following structures:

\[
\begin{align*}
\Omega_{\text{int}}(U) & \quad \Omega_{\varphi(U)} & \quad \Omega_{\mathcal{G}}(U) \\
\{a, b, c\} & \quad 1 & \quad U \\
\{a, b\} & \quad \gamma & \quad \{a, b\} \\
\{a, c\} & \quad \delta & \quad \{a, c\} \\
\{b, c\} & \quad \eta & \quad \{b\} \\
\{b\} & \quad \alpha & \quad \{c\} \\
\{c\} & \quad \beta & \quad \emptyset \\
\emptyset
\end{align*}
\]

where:

a) In \( \Omega_{\text{int}}(U) \), \( \emptyset = \text{int}(\emptyset) = \text{int}(\{a\}) \), \( \{b\} = \text{int}(\{b\}) \), \( \{c\} = \text{int}(\{c\}) \), \( \{a, b\} = \text{int}(\{a, b\}) \), \( \{b, c\} = \text{int}(\{b, c\}) \), \( \{a, c\} = \text{int}(\{a, c\}) \), \( U = \text{int}(U) \).

b) In what follows, given a set of sets \( \mathcal{X}, \mathbf{C}_{\mathcal{X}} \) stands for any combination without repetition of elements from \( \mathcal{X} \). For instance, if \( \mathcal{X} = \{\{a\}, \{b\}\} \) then \( \mathbf{C}_{\mathcal{X}} = \{\{a\}\} \) or \( \mathbf{C}_{\mathcal{X}} = \{\{b\}\} \) or \( \mathbf{C}_{\mathcal{X}} = \{\{a\}, \{b\}\} \). In \( \Omega_{\varphi(U)} \), \( 0 = \emptyset = \mathcal{A}(\emptyset) = \mathcal{A}(\{a\}) \); \( a = 0 \cup \{\{a\}, \{b\}, \{b, c\}\} = \mathcal{A}(\{\{a\}, \{b\}, \{b, c\}\}) \); \( \beta = 0 \cup \{\{c\}\} = \mathcal{A}(\{\{c\}\} \cup \mathbf{C}_0) \); \( \gamma = a \cup \{a, b\} = \mathcal{A}(\{a, b\} \cup \mathbf{C}_a) \); \( \delta = a \cup \beta = \mathcal{A}(\mathbf{C}_a \cup \mathbf{C}_\beta) \); \( \eta = \beta \cup \{a, c\} = \mathcal{A}(\{a, c\} \cup \mathbf{C}_\beta) \); \( 1 = \gamma \cup \eta \cup \{\emptyset\} = \mathcal{A}(\emptyset \cup \mathbf{C}_\gamma \cup \mathbf{C}_\eta) \).

We give some example of calculation of \( \mathcal{A} \) (we use all the canonical parentheses):

\[
\begin{align*}
\mathcal{A}(\{a\}) &= [R^-(\emptyset)](\{a\}) = [R^-](\{b\}) = \{\emptyset, \{a\}, \{b\}, \{b, c\}\}.
\mathcal{A}(\{b\}, \{c\}) &= [R^-](\{b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}\}.
\end{align*}
\]
A. $\Omega_{\text{int}}(U)$ is not closed under intersections (for instance, $\{a, b\} \cap \{a, c\} = \{a\} \notin \Omega_{\text{int}}(U)$). In fact, we need the operation $\wedge$: $\{a, b\} \wedge \{a, c\} = \text{int}(\{a, b\} \cap \{a, c\}) = \text{int}(\{a\}) = \emptyset$.

B. $\Omega_A(\emptyset(U))$ is not closed under unions (for instance, $\emptyset \cup \gamma = \emptyset \in \emptyset(U)$), which is not an element of $\Omega_A(\emptyset(U))$. Indeed, we need the operation $\vee$: $\eta \vee \gamma = A(\emptyset \cup \gamma) = 1$ (similarly, $1 = A(\emptyset \cup \delta) = A(\delta \cup \gamma)$).

C. We can easily verify the following correspondence:

<table>
<thead>
<tr>
<th>sets in $U$</th>
<th>sets in $\emptyset(U)$</th>
<th>symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>0</td>
</tr>
<tr>
<td>${b}$</td>
<td>${\emptyset, {a}, {b}, {b, c}}$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>${c}$</td>
<td>${\emptyset, {c}}$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>${a, b}$</td>
<td>${\emptyset, {a}, {b}, {a, b}, {b, c}}$</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>${a, c}$</td>
<td>${\emptyset, {c}, {a, c}}$</td>
<td>$\eta$</td>
</tr>
<tr>
<td>${b, c}$</td>
<td>${\emptyset, {a}, {b}, {c}, {b, c}}$</td>
<td>$\delta$</td>
</tr>
<tr>
<td>$U$</td>
<td>$\emptyset \cup \emptyset(U)$</td>
<td>1</td>
</tr>
</tbody>
</table>

We can therefore verify, for instance, that $\eta$ is the top element of an equivalence class modulo $\Leftrightarrow$. In fact $\emptyset(U) = \{\emptyset, \{c\}, \{a, c\}\}$ and it is the top element of the equivalence class $[\{\{a, c\}\}, \{\{c\}, \{a, c\}\}, \{\emptyset, \{a, c\}\}, \{\emptyset, \{a\}, \{a, c\}\}]$, i.e. $\{\{a, c\}\} \cup C_\beta$.

D. (left) fails: $\{a, c\} \preceq \{a, c\}$, but $\{a, c\} \cdot \{a, b\} = \{a, c\} \cap \{a, b\} = \{a\}$, and $\{a\}$ is not semicovered by $\{\{a, c\}\}$. In fact $\langle R \rangle(\{a\}) = R^\rightarrow(\{a\}) = \{b\}$, while $\langle R \rangle(\{\{a, c\}\}) = R^\rightarrow(\{a, c\}) = \{a, c\}$ (and notice that $\mathbf{N2}$ fails in the neighborhood system $\{R(x)\}_{x \in U}$).

E. (right) does not hold. Notice for instance, that $\{b, c\} \preceq \{a\}$ and $\{b, c\} \preceq \{b\}$, but $\{b, c\}$ is not semicovered by $\{\{a\}\} \cdot \{\{b\}\} = \emptyset$, because $\langle R \rangle(\{b, c\})$ (viz. $\emptyset$) is not included in $\langle R \rangle(\{\emptyset\})$ (viz. $\emptyset$) (remember that $\langle R \rangle(\{b, c\})$ is short for $\langle R \rangle(\{\{c\}\})$).

This also shows that:

F. (stability) does not hold. We show this by exhibiting a counterexample of ($A$-stability).

In fact, $A(\langle \{a\}\rangle) \cdot A(\langle \{b\}\rangle) = A(\emptyset, \{a\}, \{b\}, \{b, c\}) \cdot A(\emptyset, \{a\}, \{b\}, \{b, c\}) = \emptyset, \{a\}, \{b\}, \{b, c\})$. On the contrary, $A(\{a\} \cdot \{b\}) = A(\emptyset) = \emptyset$. 

For instance, $\gamma \cdot \eta = A(\emptyset, \{c\}, \{a\}) = [R^\subset](\{b, c\}) = \emptyset, \{c\}, \{a\}, \{b, c\} = \delta$.

It is immediate to observe that $(\Omega_A(U), \lor, \land)$ is not a distributive lattice (indeed, (stability) does not hold). Moreover, not even $\cdot$ distributes over $\lor$. For instance, $\eta \cdot (\gamma \lor \delta) = \eta \cdot 1 = \eta$, whereas $(\eta \cdot \gamma) \lor (\eta \cdot \delta) = \delta \lor \delta = \delta$. 
A logical interpretation

Pretopological formal systems provide models for limited resource logics

<table>
<thead>
<tr>
<th>Logic formula</th>
<th>Pretopological interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( A \in \Omega_{\mathcal{A}}(M) )</td>
</tr>
<tr>
<td>( a \oplus b )</td>
<td>( A \cup B ) additive conjunction</td>
</tr>
<tr>
<td>( a \otimes b )</td>
<td>( A \bullet B ) multiplicative conjunction</td>
</tr>
<tr>
<td>( a &amp; b )</td>
<td>( A \cap B ) additive conjunction</td>
</tr>
<tr>
<td>( a \vdash b )</td>
<td>( A \downarrow B )</td>
</tr>
</tbody>
</table>

\[ \frac{A \downarrow C}{A \cdot B \downarrow C} \quad \frac{\Gamma, a \vdash c}{\Gamma, a \otimes b \vdash c} \equiv \text{left } \& \]

\[ \frac{A \downarrow B \quad A \downarrow C}{A \downarrow B \cdot C} \quad \frac{\Gamma \vdash a \quad \Gamma \vdash b}{\Gamma \vdash a \otimes b} \equiv \text{right } \& \]